Quadratically connected sequences 1.
Sequences
https://www.linkedin.com/groups/8313943/8313943-6438263236520407042
The sequence $(x_n)_{\mathbb{N}}$ is given by
$x_n = \frac{1}{4}\left(\left(2+\sqrt{3}\right)^{2n-1} + \left(2-\sqrt{3}\right)^{2n-1}\right), n \in \mathbb{N}.$
Prove that each x_n equal to the sum of squares of two consecutive integers.
Solution by Arkady Alt, San Jose,California, USA.
First note that $x_n = \frac{2-\sqrt{3}}{4}\left(7+4\sqrt{3}\right)^n + \frac{2+\sqrt{3}}{4}\left(7-4\sqrt{3}\right)^n$ and, therefore, can be defined by recurrence
(1) $x_{n+1} - 14x_n + x_{n-1} = 0, n \in \mathbb{N}$
with initial conditions $x_0 = 1, x_1 = 1.$
$(x_2 = 13 = 2^2 + 3^2, x_3 = 14 \cdot 13 - 1 = 9^2 + 10^2).$
We will find a sequence (b_n) of integer numbers such that
$x_n = b_n^2 + (b_n+1)^2 \iff 2x_n - 1 = (2b_n+1)^2 \iff y_n = a_n^2$,
where $y_n := 2x_n - 1$ and $a_n := 2b_n + 1.$
By substitution $x_n = \frac{y_n+1}{2}$ in the recurrence **(1)** we obtain
$\frac{y_{n+1}+1}{2} - 14\cdot\frac{y_n+1}{2} + \frac{y_{n-1}+1}{2} = 0 \iff y_{n+1} - 14y_n + y_{n-1} - 12 = 0,$
where $y_0 = y_1 = 1$ and, therefore, $y_2 = 25.$
We will prove that a_n is defined by recurrence
(2) $a_{n+1} - 4a_n + a_{n-1} = 0,\ n \in \mathbb{N}$
with initial conditions $a_0 = -1, a_1 = 1.$ Obvious that $a_n \in \mathbb{N}.$
Note that
$(a_{n+1} + a_{n-1})^2 = 16a_n^2 \iff a_{n+1}^2 + a_{n-1}^2 + 2a_{n+1}a_{n-1} = 16a_n^2 \iff$
$a_{n+1}^2 + a_{n-1}^2 - 14a_n^2 = 2\left(a_n^2 - a_{n+1}a_{n-1}\right), a_2 = 4\cdot a_1 - a_0 = 4 + 1 = 5.$
Since $a_{n+1}^2 - a_{n+2}a_n = a_{n+1}(4a_n - a_{n-1}) - (4a_{n+1} - a_n)a_n = a_n^2 - a_{n-1}a_{n+1}$
for any $n \in \mathbb{N}$ then $a_n^2 - a_{n-1}a_{n+1} = a_1^2 - a_0a_2 = 1 + 5 = 6$ and,
therefore, $a_{n+1}^2 + a_{n-1}^2 - 14a_n^2 = 12.$
Since $a_1^2 = y_1, a_2^2 = y_2$ and both sequences $(y_n)_{n\geq1}$, $\left(a_n^2\right)_{n\geq1}$ satisfies
to the same recurrence then $y_n = a_n^2$ for any $n \in \mathbb{N}.$
By substitution $a_n = 2b_n + 1$ in the recurrence **(2)** and initial conditions
$a_0 = -1, a_1 = 1$ we obtain
$2b_{n+1} + 1 - 4(2b_n + 1) + 2b_{n-1} + 1 = 0 \iff b_{n+1} - 4b_n + b_{n-1} = 1, n \in \mathbb{N}$
and $b_0 = -1, b_1 = 0.$ And, of course $b_n,$ is integer for any $n \in \mathbb{N}$
(For example $b_2 = 4\cdot 0 - (-1) + 1 = 2, b_3 = 4\cdot 2 - 0 + 1 = 9, ..)$

Quadratically connected sequences 2.
Sequences(Posted 08.23.18)
https://www.linkedin.com/groups/8313943/8313943-6437946598717816833
The sequence $(x_n)_{n\geq1}$ is defined by $x_1 = x_2 = 1$, and
$x_{n+2} = 14x_{n+1} - x_n - 4, n > 1.$
Prove that each member of the given sequence is a perfect square.
Solution by Arkady Alt, San Jose,California, USA.

Let $(a_n)_{n\geq 1}$ be sequence defined by recurrence $a_{n+1}-4a_n+a_{n-1}=0,\ n\in\mathbb{N}$ with initial condition $a_0=3, a_1=1$. Obvious that $a_n\in\mathbb{N}$.
Since $(a_{n+1}+a_{n-1})^2=16a_n^2 \iff a_{n+1}^2+a_{n-1}^2+2a_{n+1}a_{n-1}=16a_n^2 \iff$ $a_{n+1}^2+a_{n-1}^2-14a_n^2=2\left(a_n^2-a_{n+1}a_{n-1}\right), a_2=4\cdot a_1-a_0=4-3=1$ and $a_{n+1}^2-a_{n+2}a_n=a_{n+1}\left(4a_n-a_{n-1}\right)-\left(4a_{n+1}-a_n\right)a_n=a_n^2-a_{n-1}a_{n+1}$ for any $n\in\mathbb{N}$ then $a_n^2-a_{n-1}a_{n+1}=a_1^2-a_0a_2=1-3=-2$ and, therefore, $a_{n+1}^2+a_{n-1}^2-14a_n^2=-4$.
Since $a_1^2=x_1, a_2^2=x_2$ and both sequences $(x_n)_{n\geq 1}$, $\left(a_n^2\right)_{n\geq 1}$ satisfies to the same recurrence then $x_n=a_n^2$ for any $n\in\mathbb{N}$

